

BRAID GROUP ACTIONS ON COIDEAL SUBALGEBRAS OF QUANTIZED ENVELOPING ALGEBRAS

STEFAN KOLB AND JACOPO PELLEGRINI

ABSTRACT. We construct braid group actions on coideal subalgebras of quantized enveloping algebras which appear in the theory of quantum symmetric pairs. In particular, we construct an action of the semidirect product of \mathbb{Z}^n and the classical braid group in n strands on the coideal subalgebra corresponding to the symmetric pair $(\mathfrak{sl}_{2n}(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C}))$. This proves a conjecture by Molev and Ragoucy. We expect similar actions to exist for all symmetric Lie algebras. The given actions are inspired by Lusztig's braid group action on quantized enveloping algebras and are defined explicitly on generators. Braid group and algebra relations are verified with the help of the package **Quagroup** within the computer algebra program **GAP**.

1. INTRODUCTION

In the theory of quantum groups an important role is played by Lusztig's braid group action on the quantized enveloping algebra $U_q(\mathfrak{g})$ of a complex simple Lie algebra \mathfrak{g} [Lus90], [Lus93]. This braid group action allows the definition of root vectors and Poincaré-Birkhoff-Witt bases. It is ubiquitous in the representation theory of $U_q(\mathfrak{g})$ and appeared for instance in the investigation of canonical bases [Lus96], the construction of quantum Schubert cells [CKP95], and the classification of coideal subalgebras [HS09].

Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be an involutive Lie algebra automorphism, that is $\theta^2 = \text{id}$, and let \mathfrak{k} be the Lie subalgebra of \mathfrak{g} consisting of elements fixed under θ . In a series of papers G. Letzter constructed and investigated quantum group analogs of $U(\mathfrak{k})$ as one-sided coideal subalgebras $U'_q(\mathfrak{k})$ of $U_q(\mathfrak{g})$ [Let99], [Let02]. The algebras $U'_q(\mathfrak{k})$ can be given explicitly in terms of generators and relations [Let03] and encompass various classes of quantum analogs of $U(\mathfrak{k})$ which had been constructed previously. We call the algebras $U'_q(\mathfrak{k})$ quantum symmetric pair coideal subalgebras. For \mathfrak{g} of classical type a different construction was given by Noumi and his collaborators [Nou96], [NS95], [Dij96]. In the influential paper [Nou96] Noumi constructed quantum algebras $U'_q(\mathfrak{so}_n)$ and $U'_q(\mathfrak{sp}_{2n})$ corresponding to the symmetric pairs $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$ and $(\mathfrak{sl}_{2n}(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C}))$, respectively. Even earlier the algebra $U'_q(\mathfrak{so}_n)$ had appeared in

2000 *Mathematics Subject Classification.* 17B37.

Key words and phrases. Quantized enveloping algebras, braid groups, coideal subalgebras, quantum symmetric pairs.

Both authors are very grateful to J.V. Stokman for numerous mathematical discussions and support. They also thank I. Heckenberger for verifying the G2 braid relation in Section 3 with the help of the computer algebra program FELIX. This research project started out while J. Pellegrini was an LLP/Erasmus visiting student at the University of Amsterdam in the first half of 2010. He is grateful to the Korteweg-de Vries Institute for hospitality. The work of S. Kolb was supported by the Netherlands Organization for Scientific Research (NWO) within the VIDI-project "Symmetry and modularity in exactly solvable models".

the work of Gavrilik and Klimyk [GK91]. The coideal subalgebras $U'_q(\mathfrak{so}_n)$ and $U'_q(\mathfrak{sp}_{2n})$ are special examples of quantum symmetric pair coideal subalgebras.

Recently, examples of braid group actions on quantum symmetric pair coideal subalgebras $U'_q(\mathfrak{k})$ appeared in the literature. Let $Br(\mathfrak{a}_{n-1})$ denote the classical braid group in n strands, that is, the braid group corresponding to Dynkin type A_{n-1} . Molev and Ragoucy [MR08], and independently Chekhov [Che07], constructed an action of $Br(\mathfrak{a}_{n-1})$ on $U'_q(\mathfrak{so}_n)$ by algebra automorphisms. This action is a quantum analog of the action of the symmetric group S_n on $\mathfrak{so}_n(\mathbb{C})$ by simultaneous permutation of rows and columns. By similar reasoning Molev and Ragoucy conjectured that the action of the semidirect product $(\mathbb{Z}/4\mathbb{Z})^n \rtimes S_n$ on $\mathfrak{sp}_{2n}(\mathbb{C})$ has a quantum analog. In the present paper we verify this conjecture.

Theorem 1.1. ([MR08, Conjecture 4.7]) *There exists an action of the group $\mathbb{Z}^n \rtimes Br(\mathfrak{a}_{n-1})$ on $U'_q(\mathfrak{sp}_{2n})$ by algebra automorphisms which is a quantum analog of the action of $(\mathbb{Z}/4\mathbb{Z})^n \rtimes S_n$ on $\mathfrak{sp}_{2n}(\mathbb{C})$.*

The aim of this paper is to understand the action of $Br(\mathfrak{a}_{n-1})$ on $U'_q(\mathfrak{so}_n)$ and the action of $\mathbb{Z}^n \rtimes Br(\mathfrak{a}_{n-1})$ on $U'_q(\mathfrak{sp}_{2n})$ within the general theory of quantum symmetric pairs. More specifically, let $\{\alpha_i \mid i \in I\}$ be a set of simple roots for the root system of \mathfrak{g} and let $\{s_i \mid i \in I\}$ denote the generators of the corresponding braid group $Br(\mathfrak{g})$. Recall that involutive automorphisms of \mathfrak{g} are classified in terms of pairs (X, τ) where $X \subset I$ and τ is a diagram automorphism of \mathfrak{g} of order at most two [Ara62]. We write Br_X to denote the subgroup of $Br(\mathfrak{g})$ generated by all s_i with $i \in X$. Let, moreover, Σ denote the restricted root system corresponding to θ and let $Br(\Sigma, \theta)$ denote the corresponding braid group. One can show that there exists a natural action of a semidirect product $Br_X \rtimes Br(\Sigma, \theta)$ on \mathfrak{k} . Extending Molev's and Ragoucy's original conjecture, we expect that this action has a quantum analog.

Conjecture 1.2. *There exists an action of the group $Br_X \rtimes Br(\Sigma, \theta)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms which is a quantum analog of the action of $Br_X \rtimes Br(\Sigma, \theta)$ on \mathfrak{k} .*

In the present paper we prove Conjecture 1.2 for the following three example classes.

- (I) \mathfrak{g} arbitrary, $X = \emptyset$, and $\tau = \text{id}$,
- (II) \mathfrak{g} arbitrary, $X = \emptyset$, and $\tau \neq \text{id}$,
- (III) $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$, $X = \{1, 3, 5, \dots, 2n-1\}$, and $\tau = \text{id}$,

where in case (III) we use the standard ordering of simple roots. In case (I) the involution θ coincides with the Chevalley automorphism ω of \mathfrak{g} , and in case (II) one has $\theta = \tau \circ \omega$ where τ is a nontrivial diagram automorphism. Proving Conjecture 1.2 in case (III) also proves Theorem 1.1. Indeed, in this case $Br_X = \mathbb{Z}^n$, the restricted root system Σ is of type A_{n-1} , and the quantum symmetric pair coideal subalgebra $U'_q(\mathfrak{k})$ coincides with Noumi's $U'_q(\mathfrak{sp}_{2n})$, see Remark 5.3.

The construction of the action of $Br(\Sigma, \theta)$ on $U'_q(\mathfrak{k})$ is guided by Lusztig's braid group action on $U_q(\mathfrak{g})$. There exists a natural group homomorphism

$$i_{\Sigma, \theta} : Br(\Sigma, \theta) \rightarrow Br(\mathfrak{g})$$

but $U'_q(\mathfrak{k})$ is not invariant under the Lusztig action of $i_{\Sigma, \theta}(Br(\Sigma, \theta))$. Nevertheless, in the example classes (I), (II), and (III) above, it is possible to modify the restriction of the Lusztig action of $i_{\Sigma, \theta}(Br(\Sigma, \theta))$ to $U'_q(\mathfrak{k})$ in such a way that $U'_q(\mathfrak{k})$ is

mapped to itself. To this end, following [Let03], we write the algebra $U'_q(\mathfrak{k})$ explicitly in terms of generators and relations. We then make an ansatz for the action of the generators of $Br(\Sigma, \theta)$ on the generators of $U'_q(\mathfrak{k})$. The fact that this ansatz actually defines algebra automorphisms of $U'_q(\mathfrak{k})$ which satisfy the braid relations is verified by computer calculations using de Graaf's package **Quagroup** [dG07] within the computer algebra system **GAP** [GAP08]. The need for computer calculations should not be too surprising. In Lusztig's original work [Lus90], [Lus93] the verification of the braid group action also involved long calculations, and quantum symmetric pair coideal subalgebras $U'_q(\mathfrak{k})$ tend to feature more involved relations than the quantized enveloping algebra $U_q(\mathfrak{g})$. As the list of symmetric pairs in [Ara62] is finite, one could well try to establish Conjecture 1.2 for general θ by a case by case analysis. Given the computational complexity of the examples considered in this paper, however, a general proof would be more desirable. Our results give strong evidence that the statement of Conjecture 1.2 holds.

In Chekhov's work [Che07] the algebra $U'_q(\mathfrak{so}_{n+1})$ is called the quantum A_n -algebra. It appears as a deformed algebra of geodesic functions on the Teichmüller space of a disk with n marked points on the boundary. In this setting the braid group action comes from the action of the mapping class group. The paper [Che07] also contains a second quantum algebra, called the quantum D_n -algebra. It would be interesting to know if the quantum D_n -algebra also coincides with a quantum symmetric pair coideal subalgebra and whether the natural braid group action Chekhov obtains from quantum Teichmüller theory can be related to Lusztig's braid group action.

The present paper focuses on the example classes (I), (II), and (III) and does not attempt maximal generality. In Section 2 we recall the braid group action of $B(\Sigma, \theta)$ on \mathfrak{k} and fix notation for quantum groups and quantum symmetric pairs. In Sections 3, 4, 5 we prove Conjecture 1.2 for the example classes (I), (II), (III), respectively. The **GAP**-codes used to check all relations are available for download from [WWW]. Each of Sections 3 and 4 ends with a short overview over the respective **GAP**-code. Section 3 contains the results of the second named author's master thesis [Pel10] which was written under the guidance of J.V. Stokman and the first named author.

2. PRELIMINARIES

2.1. Braid group action for symmetric pairs. Let \mathfrak{g} be a complex simple Lie algebra with Cartan subalgebra \mathfrak{h} . Let $\Phi \subset \mathfrak{h}^*$ denote the corresponding root system and fix a set $\Pi = \{\alpha_i \mid i \in I\}$ of simple roots. Write W to denote the Weyl group generated by all reflections s_{α_i} for $i \in I$. Let (\cdot, \cdot) denote the W -invariant scalar product on the real vector space spanned by Φ such that all short roots α satisfy $(\alpha, \alpha) = 2$. As usual, let $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ denote the entries of the Cartan matrix of \mathfrak{g} . For $i, j \in I$ let m_{ij} denote the order of $s_{\alpha_i} s_{\alpha_j}$ in W . Let $Br(\mathfrak{g})$ denote the Artin braid group corresponding to W . More explicitly, $Br(\Sigma)$ is generated by elements $\{s_i \mid i \in I\}$ and relations

$$(2.1) \quad \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij} \text{ factors}}.$$

The braid group $Br(\mathfrak{g})$ acts on \mathfrak{g} by Lie algebra automorphisms. Let $\{e_i, f_i, h_i \mid i \in I\}$ be a set of Chevalley generators for \mathfrak{g} . For $i \in I$ define

$$(2.2) \quad \text{Ad}(s_i) = \exp(\text{ad}(e_i)) \exp(\text{ad}(-f_i)) \exp(\text{ad}(e_i))$$

where the symbol ad denotes the adjoint action and where \exp is the exponential series which is well defined on nilpotent elements. By [Ste68, Lemma 56] there exists a group homomorphism

$$(2.3) \quad \text{Ad} : Br(\mathfrak{g}) \rightarrow \text{Aut}(\mathfrak{g})$$

such that $\text{Ad}(s_i)$ is given by (2.2).

Let now $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be an involutive Lie algebra automorphism and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition into the +1 and the -1 eigenspace of θ , that is $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$. In this paper we consider the following three classes of examples.

- (I) Let \mathfrak{g} be arbitrary and let θ be the Chevalley automorphism $\omega \in \text{Aut}(\mathfrak{g})$ defined by

$$(2.4) \quad \omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega|_{\mathfrak{h}} = -\text{id}_{\mathfrak{h}}.$$

In this case, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ then $\mathfrak{k} \cong \mathfrak{so}_n(\mathbb{C})$.

- (II) Let \mathfrak{g} be arbitrary and let $\theta = \tau \circ \omega$ for a nontrivial diagram automorphism τ of order 2. A nontrivial diagram automorphism only exists if \mathfrak{g} is of type A_n for $n \geq 2$, of type D_n for $n \geq 4$, or of type E_6 .
- (III) Let $m \in \mathbb{N}$ and $\mathfrak{g} = \mathfrak{sl}_{2m}(\mathbb{C})$ with the standard ordering of the simple roots. We consider $\theta = \text{Ad}(w_X) \circ \omega$ where $w_X = s_1 s_3 s_5 \cdots s_{2m-1}$. In this case $\mathfrak{k} \cong \mathfrak{sp}_{2m}(\mathbb{C})$.

Any diagram automorphism τ for \mathfrak{g} yields a group automorphism of $Br(\mathfrak{g})$ which we denote by the same symbol τ . On the generators of $Br(\mathfrak{g})$ one has $\tau(s_i) = s_{\tau(i)}$. Now define

$$Br(\mathfrak{g}, \theta) = \begin{cases} Br(\mathfrak{g}) & \text{if } (\mathfrak{g}, \theta) = (\mathfrak{g}, \omega) \quad \text{(I)}, \\ \{b \in Br(\mathfrak{g}) \mid \tau(b) = b\} & \text{if } (\mathfrak{g}, \theta) = (\mathfrak{g}, \tau \circ \omega) \quad \text{(II)}, \\ \{b \in Br(\mathfrak{g}) \mid w_X b = b w_X\} & \text{if } (\mathfrak{g}, \theta) = (\mathfrak{sl}_{2m}(\mathbb{C}), \text{Ad}(w_X) \circ \omega) \quad \text{(III)}. \end{cases}$$

Lemma 2.1. *Under the action Ad , the subgroup $Br(\mathfrak{g}, \theta)$ of $Br(\mathfrak{g})$ maps \mathfrak{k} to itself.*

Proof. One first observes that ω commutes with $\text{Ad}(s_i)$ for all $i \in I$. In case (II) with $\theta = \tau \circ \omega$ one verifies that $\theta(\text{Ad}(b)(x)) = \text{Ad}(\tau(b))(\theta(x))$ holds for all $b \in Br(\mathfrak{g})$, $x \in \mathfrak{g}$. Similarly, in case (III) one has $\theta(\text{Ad}(b)(x)) = \text{Ad}(w_X b w_X^{-1})(\theta(x))$. The above observations imply that for all $b \in Br(\mathfrak{g}, \theta)$ one has $\text{Ad}(b) \circ \theta = \theta \circ \text{Ad}(b)$. Hence for $b \in Br(\mathfrak{g}, \theta)$ one has $\text{Ad}(b)(\mathfrak{k}) = \mathfrak{k}$. \square

In the following we use the standard ordering of simple roots as in [Bou02]. In case (II), that is for $\theta = \tau \circ \omega$, we need to distinguish three different cases.

(IIA) $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{sl}_{n+1}(\mathbb{C})$ and $\tau(i) = n - i + 1$.

(IID) $\mathfrak{g} = \mathfrak{d}_{n+1} = \mathfrak{so}_{2n+2}(\mathbb{C})$ and

$$\tau(i) = \begin{cases} i & i \neq n, n+1, \\ n & i = n+1, \\ n+1 & i = n. \end{cases}$$

(IIE) $\mathfrak{g} = \mathfrak{e}_6$ and τ is the nontrivial diagram automorphism

$$\tau(1) = 6, \tau(2) = 2, \tau(3) = 5, \tau(4) = 4, \tau(5) = 3, \tau(6) = 1.$$

Now define for each of the above cases a braid group $Br(\Sigma, \theta)$ as follows

$$\begin{aligned} \text{in case (I):} & \quad Br(\Sigma, \omega) = Br(\mathfrak{g}), \\ \text{in case (IIA) with } n = 2r: & \quad Br(\Sigma, \tau \circ \omega) = Br(\mathfrak{b}_r), \\ \text{in case (IIA) with } n = 2r-1: & \quad Br(\Sigma, \tau \circ \omega) = Br(\mathfrak{b}_r), \\ \text{in case (IID):} & \quad Br(\Sigma, \tau \circ \omega) = Br(\mathfrak{b}_n), \\ \text{in case (IIE):} & \quad Br(\Sigma, \tau \circ \omega) = Br(\mathfrak{f}_4), \\ \text{in case (III):} & \quad Br(\Sigma, \text{Ad}(w_X) \circ \omega) = Br(\mathfrak{a}_{m-1}). \end{aligned}$$

In the following we will consider the braid groups $Br(\mathfrak{g})$ and $Br(\Sigma, \theta)$ simultaneously. To avoid confusion we denote the generators of $Br(\Sigma, \theta)$ by \bar{s}_i as opposed to the notation s_i for the generators of $Br(\mathfrak{g})$.

Proposition 2.2. *There exists a group homomorphism*

$$i_{\Sigma, \theta} : Br(\Sigma, \theta) \rightarrow Br(\mathfrak{g}, \theta)$$

determined in each of the cases (I), (II), and (III) as follows:

$$\begin{aligned} \text{(I)} \quad & Br(\mathfrak{g}) \rightarrow Br(\mathfrak{g}), \quad \bar{s}_i \mapsto s_i, \\ \text{(IIA)} \quad & \text{If } n = 2r: \end{aligned}$$

$$Br(\mathfrak{b}_r) \rightarrow Br(\mathfrak{a}_{2r}), \quad \bar{s}_i \mapsto \begin{cases} s_i s_{n-i+1} & i \neq r, \\ s_r s_{r+1} s_r & i = r. \end{cases}$$

If $n = 2r-1$:

$$Br(\mathfrak{b}_r) \rightarrow Br(\mathfrak{a}_{2r-1}), \quad \bar{s}_i \mapsto \begin{cases} s_i s_{n-i+1} & i \neq r, \\ s_r & i = r. \end{cases}$$

$$\text{(IID)} \quad Br(\mathfrak{b}_n) \rightarrow Br(\mathfrak{d}_{n+1}), \quad \bar{s}_i \mapsto \begin{cases} s_i & i \neq n, \\ s_n s_{n+1} & i = n. \end{cases}$$

$$\text{(IIE)} \quad Br(\mathfrak{f}_4) \rightarrow Br(\mathfrak{e}_6), \quad \bar{s}_1 \mapsto s_1 s_6, \bar{s}_2 \mapsto s_3 s_5, \bar{s}_3 \mapsto s_4, \bar{s}_4 \mapsto s_2.$$

$$\text{(III)} \quad Br(\mathfrak{a}_{m-1}) \rightarrow Br(\mathfrak{a}_{2m-1}), \quad \bar{s}_i \mapsto s_{2i} s_{2i-1} s_{2i+1} s_{2i}.$$

Proof. The images of the generators \bar{s}_i under $i_{\Sigma, \theta}$ do indeed lie in $Br(\mathfrak{g}, \theta)$. It is verified by direct computation that the elements $i_{\Sigma, \theta}(\bar{s}_i)$ satisfy the braid relations of $Br(\Sigma, \theta)$. \square

Corollary 2.3. *In any of the cases (I), (II), and (III) there exists an action of $Br(\Sigma, \theta)$ on \mathfrak{k} by Lie algebra automorphisms. This action is given by the composition of the map $i_{\Sigma, \theta}$ from Proposition 2.2 with the action of $Br(\mathfrak{g})$ on \mathfrak{g} .*

Remark 2.4. Let Σ be the restricted root system corresponding to the symmetric Lie algebra (\mathfrak{g}, θ) , see [Ara62, 2.4]. The Dynkin diagram of Σ is given by the third column of the table in [Ara62, p. 32/33], however, Σ may be non-reduced. The braid group $Br(\Sigma, \theta)$ defined above for special examples is exactly the braid group corresponding to the root system Σ . An action of $Br(\Sigma, \theta)$ on \mathfrak{k} , generalizing the action of the above corollary, exists for any symmetric Lie algebra (\mathfrak{g}, θ) .

Remark 2.5. To compare the classical and the quantum situation for case (III) in Section 5.1 we make the action of $Br(\mathfrak{a}_{m-1})$ on $\mathfrak{sp}_{2m}(\mathbb{C})$ more explicit. Let e_i, f_i, h_i for $i = 1, \dots, 2m-1$ denote the standard Chevalley generators of $\mathfrak{sl}_{2m}(\mathbb{C})$. Define a $(2m \times 2m)$ -matrix S by

$$S = \begin{pmatrix} J & 0 & \dots & 0 \\ 0 & J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any $x \in \mathfrak{sl}_{2m}(\mathbb{C})$ one has $\theta(x) = -\text{Ad}(S)(x^t)$. The Chevalley generators e_i, f_i, h_i for odd i are invariant under θ . Define elements $b_{2j} = f_{2j} + \theta(f_{2j})$ for $j = 1, 2, 3, \dots, m-1$. Using the weight decomposition of $\mathfrak{g} = \mathfrak{sl}_{2m}$ one shows that the Lie algebra $\mathfrak{k} \cong \mathfrak{sp}_{2m}(\mathbb{C})$ is generated by the elements

$$(2.5) \quad e_i, f_i, h_i \quad \text{for } i = 1, 3, 5, \dots, 2m-1,$$

$$(2.6) \quad b_{2j} \quad \text{for } j = 1, 2, 3, \dots, m-1.$$

For any ring R and $s \in \mathbb{N}$ let $\text{Mat}_s(R)$ denote the set of $(s \times s)$ -matrices with entries in R . In view of the special form of S it is natural to consider elements in $\mathfrak{k} \cong \mathfrak{sp}_{2m}(\mathbb{C})$ as elements in $\text{Mat}_m(\text{Mat}_2(\mathbb{C}))$. The action $Br(\mathfrak{a}_{m-1})$ on \mathfrak{k} then factors through the natural action of the symmetric group S_m on $\text{Mat}_m(\text{Mat}_2(\mathbb{C}))$ by simultaneous permutations of rows and columns. We define $b_j = f_j$ for $j = 1, 3, 5, \dots, 2m-1$ and calculate

$$(2.7) \quad \text{Ad}(s_{2i}s_{2i-1}s_{2i+1}s_{2i})(b_j) = \begin{cases} [[b_{2i-2}, b_{2i-1}], b_{2i}] & \text{if } j = 2i-2, \\ b_{2i+1} & \text{if } j = 2i-1, \\ b_j & \text{if } j = 2i \text{ or } |j-2i| > 2, \\ b_{2i-1} & \text{if } j = 2i+1, \\ [[b_{2i+2}, b_{2i+1}], b_{2i}] & \text{if } j = 2i+2. \end{cases}$$

In Section 5.1 we will construct a quantum group analog of the above action. Now consider odd $j = 1, 3, 5, \dots, 2m-1$ and observe that the subspace \mathfrak{k} is invariant under the action of $\text{Ad}(s_j)$. One has $\text{Ad}(s_j^2)(f_{j+1}) = -f_{j+1}$ and $\text{Ad}(s_j)$ commutes with θ for odd j . Hence $\text{Ad}(s_j^2)(b_{j+1}) = -b_{j+1}$ and the action of s_j on $\mathfrak{sp}_{2m}(\mathbb{C})$ has order four. In other words, the operators $\text{Ad}(s_j)$ for $j = 1, 3, 5, \dots, 2m-1$ give an action of $(\mathbb{Z}/4\mathbb{Z})^m$ on $\mathfrak{sp}_{2m}(\mathbb{C})$ by Lie algebra automorphisms. Taking into account the action of S_m discussed above, one obtains the desired action of $(\mathbb{Z}/4\mathbb{Z})^m \rtimes S_m$ on $\mathfrak{sp}_{2m}(\mathbb{C})$.

2.2. Quantum groups. Let k be a field and let $q \in k \setminus \{0\}$ be not a root of unity. For technical reasons which will become apparent in Sections 4 and 5 we assume that k contains a square root $q^{1/2}$ of q . We consider the quantized enveloping algebra $U_q(\mathfrak{g})$ as the k -algebra with generators E_i, F_i, K_i, K_i^{-1} for all $i \in I$ and relations given in [Jan96, 4.3]. Recall that $U_q(\mathfrak{g})$ is a Hopf algebra with coproduct Δ determined by

$$(2.8) \quad \begin{aligned} \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \end{aligned}$$

for all $i \in I$. For any $i \in I$ one defines $q_i = q^{(\alpha_i, \alpha_i)/2}$ and for $n \in \mathbb{N}$ the q -number

$$(2.9) \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and the q -factorial $[n]_i! = [n]_i[n-1]_i \dots [2]_i$. If $(\alpha_i, \alpha_i) = 2$ then we will also write $[n]$ and $[n]!$ instead of $[n]_i$ and $[n]_i!$, respectively. As observed by Lusztig [Lus90], the action of $Br(\mathfrak{g})$ on \mathfrak{g} by Lie algebra automorphisms deforms to an action of $Br(\mathfrak{g})$ on $U_q(\mathfrak{g})$ by algebra automorphisms. The image of the generator $s_i \in Br(\mathfrak{g})$ under this action is the Lusztig automorphism T_i as given in [Jan96, 8.14]. In the following it is sometimes more convenient to work with the inverse of T_i which we therefore recall explicitly. One has

$$(2.10) \quad T_i^{-1}(E_i) = -K_i^{-1}F_i, \quad T_i^{-1}(F_i) = -E_iK_i, \quad T_i^{-1}(K_i) = K_i^{-1},$$

and

$$(2.11) \quad \begin{aligned} T_i^{-1}(K_j) &= K_j K_i^{-a_{ij}}, \\ T_i^{-1}(E_j) &= \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} E_i^{(s)} E_j E_i^{(-a_{ij}-s)}, \\ T_i^{-1}(F_j) &= \sum_{s=0}^{-a_{ij}} (-1)^s q_i^s F_i^{(-a_{ij}-s)} F_j F_i^{(s)}. \end{aligned}$$

for all $j \neq i$ where $E_i^{(n)} = \frac{E_i^n}{[n]_i!}$ and $F_i^{(n)} = \frac{F_i^n}{[n]_i!}$ for any $n \in \mathbb{N}$. Observe that if $a_{ij} = -2$ or $a_{ij} = -3$ then $(\alpha_i, \alpha_i) = 2$. Hence in these cases one may replace q_i by q in the above formulas.

2.3. Quantum symmetric pairs. For each involutive automorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ a q -analog of $U(\mathfrak{k})$ was constructed by G. Letzter [Let99], [Let02] as a one-sided coideal subalgebra $U'_q(\mathfrak{k})$ of $U_q(\mathfrak{g})$. Here we choose to work with right coideal subalgebras, that is $\Delta(U'_q(\mathfrak{k})) \subset U'_q(\mathfrak{k}) \otimes U_q(\mathfrak{g})$. In the following sections we will give the algebra $U'_q(\mathfrak{k})$ as a subalgebra of $U_q(\mathfrak{g})$ for each of the three example classes (I), (II), and (III). Our conventions slightly differ from those in [Let99], but all results from Letzter's papers translate into the present setting. In particular we will recall the presentation of $U'_q(\mathfrak{k})$ in terms of generators and relations following [Let03]. For each example class we will construct the desired action of $Br(\Sigma, \theta)$ on $U'_q(\mathfrak{k})$ by algebra automorphism. For classical \mathfrak{g} , quantum analogs of $U(\mathfrak{k})$ were previously constructed by Noumi and his coworkers [Nou96], [NS95], [Dij96]. The relations between the two approaches to quantum symmetric pairs are fairly well understood [Let99, Section 6], [Kol08].

3. THE CHEVALLEY INVOLUTION

All through this section we consider the case where \mathfrak{g} is arbitrary but θ coincides with the Chevalley involution ω . In this case, by definition, $U'_q(\mathfrak{k})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$B_i = F_i - K_i^{-1}E_i \quad \text{for all } i \in I.$$

It follows from (2.8) that $U'_q(\mathfrak{k})$ is a right coideal subalgebra of $U_q(\mathfrak{g})$. Up to slight conventional changes the following result is contained in [Let03, Theorem 7.1].

Recall that the q -binomial coefficient is defined for any $i \in I$ and any $a, n \in \mathbb{Z}$ with $n > 0$ by

$$\begin{bmatrix} a \\ n \end{bmatrix}_i = \frac{[a]_i [a-1]_i \cdots [a-n+1]_i}{[n]_i [n-1]_i \cdots [1]_i}.$$

Proposition 3.1. *The algebra $U'_q(\mathfrak{k})$ is generated over k by elements $\{B_i \mid i \in I\}$ subject only to the relations*

$$\begin{aligned} & \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_i B_i^{1-a_{ij}-s} B_j B_i^s \\ &= \begin{cases} 0 & \text{if } a_{ij} = 0, \\ -q_i^{-1} B_j & \text{if } a_{ij} = -1, \\ -q^{-1} [2]^2 (B_i B_j - B_j B_i) & \text{if } a_{ij} = -2, \\ -q^{-1} ([3]^2 + 1) (B_i^2 B_j + B_j B_i^2) + \\ \quad + q^{-1} [2] ([2][4] + q^2 + q^{-2}) B_i B_j B_i - q^{-2} [3]^2 B_j & \text{if } a_{ij} = -3. \end{cases} \end{aligned}$$

3.1. Braid group action corresponding to ω . We now construct the action of $Br(\Sigma, \theta) = Br(\mathfrak{g})$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. For $i, j \in I$ the element $T_i(B_j)$ does in general not belong to $U'_q(\mathfrak{k})$. This was already noted in [MR08]. However, the Lusztig action still serves as a guide to the construction of the desired braid group action on $U'_q(\mathfrak{k})$. In our conventions it is slightly easier to work with the inverses of the Lusztig automorphisms given by (2.10), (2.11). The general strategy, which will also be applied to the example classes (II) and (III) in the subsequent subsections, is as follows. For any $i, j \in I$ we construct an element $\tau_i^-(B_j)$ in $U'_q(\mathfrak{k})$ which coincides with $T_i^{-1}(B_j)$ up to terms of higher weight with respect to the left adjoint action of $U^0 = k\langle K_l^{\pm 1} \mid l \in I \rangle$ on $U_q(\mathfrak{g})$. In other words, $\tau_i^-(B_j)$ and $T_i^{-1}(B_j)$ have identical terms containing maximal powers of the generators F_l , $l \in I$, maybe up to a factor. For fixed i we verify that the elements $\tau_i^-(B_j)$, for all $j \in I$, define an algebra endomorphism τ_i^- of $U'_q(\mathfrak{k})$. An inverse is constructed using T_i instead of T_i^{-1} . It is then checked that the algebra automorphisms τ_i^- , $i \in I$, indeed satisfy the braid relations of $Br(\Sigma, \theta)$.

More precisely, for any $i, j \in I$ define

$$(3.1) \quad \tau_i^-(B_j) = \begin{cases} B_j & \text{if } j=i \text{ or } a_{ij}=0, \\ B_i B_j - q_i B_j B_i & \text{if } a_{ij} = -1, \\ [2]^{-1} (B_i^2 B_j - q[2] B_i B_j B_i + q^2 B_j B_i^2) + B_j & \text{if } a_{ij} = -2, \\ [3]^{-1} [2]^{-1} (B_i^3 B_j - q[3] B_i^2 B_j B_i + q^2 [3] B_i B_j B_i^2 \\ \quad - q^3 B_j B_i^3 + q^{-1} (B_i B_j - q^3 B_j B_i)) \\ \quad + (B_i B_j - q B_j B_i) & \text{if } a_{ij} = -3, \end{cases}$$

One calculates

$$(3.2) \quad T_i^{-1}(B_j) = \tau_i^-(B_j) + \epsilon(a_{ij})$$

where

$$\begin{aligned}
\epsilon(2) &= (K_j^{-1} - q_j^{-2} K_j) E_j, \\
\epsilon(0) &= 0, \\
\epsilon(-1) &= (q_i - q_i^{-1}) F_j K_i^{-1} E_i, \\
\epsilon(-2) &= -(q - q^{-1}) (q^{-1} F_2 K_3^{-2} + (q^2 - 1) F_2 K_3^{-2} E_3^2 + (F_3 F_2 - q^2 F_2 F_3) K_3^{-1} E_3), \\
\epsilon(-3) &= -(q - q^{-1}) \left[\left(\frac{1}{[2]} F_1^2 F_2 - q^2 F_1 F_2 F_1 + \frac{q^4}{[2]} F_2 F_1^2 \right) K_1^{-1} E_1 \right. \\
&\quad + (F_1 F_2 - q^3 F_2 F_1) (q^{-1} K_1^{-2} + (q^2 - 1) K_1^{-2} E_1^2) \\
&\quad \left. + q^{-1} (q^3 - q^{-3}) F_2 K_1^{-3} E_1 + q^3 (q - q^{-1})^2 F_2 K_1^{-3} E_1^3 \right].
\end{aligned}$$

The formulas for $\epsilon(-2)$ and $\epsilon(-3)$ are most easily verified by **GAP**-computations, see Subsection 3.2.

Remark 3.2. Let U^+ denote the subalgebra of $U_q(\mathfrak{g})$ generated by all E_i with $i \in I$. By [Let99, Theorem 2.4] the multiplication map

$$m : U'_q(\mathfrak{k}) \otimes U^0 \otimes U^+ \rightarrow U_q(\mathfrak{g})$$

is an isomorphism of vector spaces. Let $\pi^0 : U^0 \rightarrow k$ map any Laurent polynomial in $U^0 = k\langle K_i, K_i^{-1} \mid i \in I \rangle$ to its constant term. One defines a projection of vector spaces $\pi : U_q(\mathfrak{g}) \rightarrow U'_q(\mathfrak{k})$ by $\pi(u) = (\text{id} \otimes \pi^0 \otimes \varepsilon) \circ m^{-1}(u)$. It follows from the above formulas that

$$\tau_i^-(B_j) = \pi \circ T_i^{-1}(B_j).$$

However, the projection map π is no algebra homomorphism. Therefore it is a priori unclear that τ_i^- is an algebra homomorphism. Nevertheless, this holds by the following Theorem.

Theorem 3.3. *Let $i \in I$.*

- 1) *There exists a unique algebra automorphism τ_i^- of $U'_q(\mathfrak{k})$ such that $\tau_i^-(B_j)$ is given by (3.1).*
- 2) *The inverse automorphism τ_i of τ_i^- is determined by*

$$(3.3) \quad \tau_i(B_j) = \begin{cases} B_j & \text{if } j=i \text{ or } a_{ij}=0, \\ B_j B_i - q_i B_i B_j & \text{if } a_{ij} = -1, \\ [2]^{-1} (B_j B_i^2 - q[2] B_i B_j B_i + q^2 B_i^2 B_j) + B_j & \text{if } a_{ij} = -2, \\ [3]^{-1} [2]^{-1} (B_j B_i^3 - q[3] B_i B_j B_i^2 + q^2 [3] B_i^2 B_j B_i \\ \quad - q^3 B_i^3 B_j + q^{-1} (B_j B_i - q^3 B_i B_j)) \\ \quad + (B_j B_i - q B_i B_j) & \text{if } a_{ij} = -3. \end{cases}$$

- 3) *There exists a unique group homomorphism $Br(\mathfrak{g}) \rightarrow \text{Aut}_{alg}(U'_q(\mathfrak{k}))$ such that $s_j \mapsto \tau_j^-$ for all $j \in I$.*

The proof of the theorem is given by direct computations using the computer algebra package **QuaGroup** [dG07] within **GAP** [GAP08] for calculations with quantum enveloping algebras. More details will be given in Subsection 3.2. For \mathfrak{g} of type *ADE*, however, the statement of Theorem 3.3 follows from results in [MR08] as explained in the following two remarks.

Remark 3.4. Assume that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. We want to relate the above theorem to the braid group action constructed in [MR08]. To this end define $S_i = (q - q^{-1})B_i$ for $i = 1, \dots, N-1$ and observe that by Proposition 3.1 the algebra $U'_q(\mathfrak{k})$ is generated by the elements S_i subject only to the relations given in [MR08, above Theorem 2.1]. Hence, in this case, $U'_q(\mathfrak{k})$ coincides with the algebra $U'_q(\mathfrak{o})$ considered in [MR08] and originally introduced by Gavrilik and Klimyk in [GK91]. By [MR08, Theorem 2.1], for any $i = 1, \dots, N-1$, there exists an automorphism β_i of $U'_q(\mathfrak{k})$ such that

$$\beta_i(B_j) = \begin{cases} -[B_i, B_j]_q & \text{if } j = i+1, \\ [B_i, B_j]_q & \text{if } j = i-1, \\ -B_i & \text{if } i = j, \\ B_j & \text{else} \end{cases}$$

where by definition $[a, b]_q = ab - qba$ for any $a, b \in U_q(\mathfrak{g})$. Now we observe that $\tau_i^- = \beta_i \circ \kappa_i$ where $\kappa_i : U'_q(\mathfrak{k}) \rightarrow U'_q(\mathfrak{k})$ is the algebra automorphism defined by

$$\kappa_i(B_j) = \begin{cases} -B_j & \text{if } j = i \text{ or } j = i+1, \\ B_j & \text{else.} \end{cases}$$

In view of commutation relations

$$\begin{aligned} \kappa_{i+1} \circ \beta_i &= \beta_i \circ \kappa_{i+1}, & \kappa_{i+1} \circ \beta_{i+1} &= \beta_{i+1} \circ \kappa_i, \\ \kappa_i \circ \beta_{i+1} &= \beta_{i+1} \circ \kappa_{i+1} \circ \kappa_{i+2} \circ \dots \circ \kappa_{N-1} \end{aligned}$$

the braid relations for the automorphisms $\{\beta_i \mid i \in I\}$ are equivalent to the braid relations for the automorphisms $\{\tau_i^- \mid i \in I\}$. Hence the statements of Theorem 3.3 1) and 3) for $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$ are equivalent to [MR08, Theorem 2.1] the proof of which also contains a formula for the inverse of β_i .

Remark 3.5. Assume now that \mathfrak{g} is of type D_n or E_n . The fact that τ_i^- is an algebra endomorphism of $U'_q(\mathfrak{k})$ follows from the corresponding fact for \mathfrak{g} of type A_n because the elements $\tau_i^-(B_j)$ and $\tau_i^-(B_k)$ are contained in the subalgebra of $U_q(\mathfrak{g})$ corresponding the subset $\{i, j, k\}$ of I . The same holds for the inverse τ_i and therefore τ_i^- is an algebra automorphism. Each side of a braid relation evaluated on any generator B_k is again contained in the subalgebra of $U_q(\mathfrak{g})$ corresponding to a subset of I with at most three elements. Hence the braid relations for \mathfrak{g} of type A_n also imply the braid relations for $\{\tau_i^- \mid i \in I\}$ if \mathfrak{g} is of type D_n or E_n .

3.2. The proof of Theorem 3.3. In this subsection we explain how to verify Theorem 3.3 using the package `QuaGroup` under `GAP`. In view of Remarks 3.4 and 3.5 it remains to consider the cases where \mathfrak{g} is of type B_n, C_n, F_4 , or G_2 . By Theorem 3.3 for type A_n it even suffices to consider the cases where \mathfrak{g} is of type B_3, C_3, F_4 , or G_2 . Moreover, the claim of Theorem 3.3 for \mathfrak{g} of type F_4 will follow from the corresponding claims for \mathfrak{g} of type B_3 and C_3 . Hence we only need to consider the three remaining cases B_3, C_3 , and G_2 .

In the cases B_3 and C_3 this is done by the `GAP` codes `I-B3.txt` and `I-C3.txt` which are available from [WWW]. In each code the generators of $U'_q(\mathfrak{k})$ are defined and it is checked that the relations of Proposition 3.1 are satisfied. Next it is verified that for fixed i the images $\tau_i^-(B_j)$ and $\tau_i(B_j)$ also satisfy the relations of Proposition 3.1. This proves that Equations (3.1) and (3.3) give well-defined algebra endomorphisms of $U'_q(\mathfrak{k})$. It is then checked that $\tau_i^- \circ \tau_i(B_j) = B_j = \tau_i \circ \tau_i^-(B_j)$

for all i, j which implies that τ_i and τ_i^- are mutually inverse and hence algebra automorphisms. Finally, the braid relations are verified when evaluated on the generators. This completes the proof of Theorem 3.3 in these cases.

In the case G_2 parts 1) and 2) of Theorem 3.3 are verified in the same way as described above using the `GAP` code `I-G2.txt` [WWW]. However, due to memory problems, the `GAP` code provided in `I-G2.txt` crashes when it tries to verify the G_2 braid relations. Istvan Heckenberger kindly checked the G_2 braid relations for us, using the noncommutative algebra program `FELIX` [AK]. His code `G2-Braid.flx` and his output file `G2-Braid.aus` are also available from [WWW]. In these calculations it turned out that $(\tau_1 \circ \tau_2)^3 = \text{id}_{U'_q(\mathfrak{k})} = (\tau_2 \circ \tau_1)^3$ if \mathfrak{g} is of type G_2 . Analogously, one has $(\tau_1 \circ \tau_2)^2 = \text{id}_{U'_q(\mathfrak{k})} = (\tau_2 \circ \tau_1)^2$ if \mathfrak{g} is of type B_2 , however not in higher rank.

4. THE INVOLUTIVE AUTOMORPHISM $\theta = \tau \circ \omega$

In this section we consider the case (II), that is \mathfrak{g} is of type A_n , D_n , or E_6 and $\theta = \tau \circ \omega$ where τ is the nontrivial diagram automorphism of order 2. In this case, by definition, $U'_q(\mathfrak{k})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$(4.1) \quad B_i = F_i - K_i^{-1} E_{\tau(i)}, \quad K_i K_{\tau(i)}^{-1}$$

for all $i \in I$. Again it follows from (2.8) that $U'_q(\mathfrak{k})$ is a right coideal subalgebra of $U_q(\mathfrak{g})$. Let kT_θ denote the subalgebra of $U'_q(\mathfrak{k})$ generated by the elements $K_i K_{\tau(i)}^{-1}$ for all $i \in I$ and observe that kT_θ is a Laurent polynomial ring. The following result is contained in [Let03, Theorem 7.1].

Proposition 4.1. *The algebra $U'_q(\mathfrak{k})$ is generated over kT_θ by elements $\{B_i \mid i \in I\}$ subject only to the relations*

$$(4.2) \quad \begin{aligned} K_i K_{\tau(i)}^{-1} B_j &= q^{(\alpha_j, \alpha_{\tau(i)} - \alpha_i)} B_j K_i K_{\tau(i)}^{-1} && \text{for all } i, j \in I, \\ B_i B_j - B_j B_i &= \delta_{\tau(i), j} \frac{K_i K_{\tau(i)}^{-1} - K_{\tau(i)} K_i^{-1}}{q - q^{-1}} && \text{if } a_{ij} = 0, \\ B_i^2 B_j - (q + q^{-1}) B_i B_j B_i + B_j B_i^2 &= -\delta_{i, \tau(i)} q^{-1} B_j - \\ &\quad -\delta_{j, \tau(i)} (q + q^{-1}) B_i (q^{-1} K_i K_{\tau(i)}^{-1} + q^2 K_{\tau(i)} K_i^{-1}) && \text{if } a_{ij} = -1. \end{aligned}$$

Remark 4.2. In the case (IIA) with $n = 2r - 1$ the theory of quantum symmetric pairs actually provides a family of coideal subalgebras $U'_q(\mathfrak{k})_s$ depending on a parameter $s \in k$. By definition, $U'_q(\mathfrak{k})_s$ is the subalgebra generated by the elements (4.1) for $i \neq r$ and by $B_r = F_r - K_r^{-1} E_r + s K_r^{-1}$. By [Let03, Theorem 7.1], however, the $U'_q(\mathfrak{k})_s$ are pairwise isomorphic as algebras for different parameters s . For the purpose of constructing an action of $Br(\mathfrak{g}, \theta)$ on $U'_q(\mathfrak{k})_s$ by algebra automorphisms it hence suffices to consider the case $s = 0$ only.

The case $a_{ij} = -1$ and $j = \tau(i)$, which leads to an additional summand in the last relation in Proposition 4.1, can only occur if \mathfrak{g} is of type A_n with even n . For simplicity we first exclude this case in the following subsection. It will be treated in Subsection 4.2. Subsections 4.3 and 4.4 are devoted to the braid group actions on $U'_q(\mathfrak{k})$ in the cases (IID) and (IIE), respectively. All theorems in the present section are verified by `GAP` calculations. More details and references to the `GAP`-codes are given in Subsection 4.5.

4.1. The braid group action in the case (IIA) for odd n . Throughout this subsection we consider the case (IIA) with $n = 2r - 1$. By Proposition 2.2 and Corollary 2.3 we aim to construct an action of the braid group $Br(\mathfrak{b}_r)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. Hence we need to construct a family of algebra automorphisms $\{\tau_1, \dots, \tau_r\}$ of $U'_q(\mathfrak{k})$ which satisfy the type B_r braid relations. Again, the construction of the τ_i is guided by the Lusztig automorphisms of $U_q(\mathfrak{g})$. For $i = 1, \dots, r - 1$ and $x \in kT_\theta$ define elements $\tau_i(x), \tau_i^-(x) \in kT_\theta$ by

$$(4.3) \quad \tau_i(x) = \tau_i^-(x) = T_i T_{\tau(i)}(x)$$

and define

$$(4.4) \quad \tau_r(x) = \tau_r^-(x) = T_r(x).$$

Observe that $\tau_r(x) = x$ for all $x \in kT_\theta$. It remains to define the action of $\{\tau_i \mid i = 1, \dots, r\}$ on the generators $\{B_j \mid j = 1, \dots, n\}$ of $U'_q(\mathfrak{k})$. The commutator relations (4.2) together with (4.3) already impose a significant restriction. For $1 \leq i \leq r - 1$ and $j = i - 1$ one can check that

$$[B_i, B_j]_q [B_{\tau(i)}, B_{\tau(j)}]_q - [B_{\tau(i)}, B_{\tau(j)}]_q [B_i, B_j]_q = q \frac{\tau_i^-(K_j K_{\tau(j)}^{-1}) - \tau_j^-(K_{\tau(j)} K_j^{-1})}{q - q^{-1}}.$$

In view of Equation (4.2) the additional factor q on the right hand side explains the appearance of fractional powers of q in the following definition. For $1 \leq i \leq r - 1$ define

$$(4.5) \quad \tau_i^-(B_j) = \begin{cases} q^{-1/2} [B_i, B_j]_q & \text{if } a_{ij} = -1 \text{ and } a_{\tau(i)j} \neq -1, \\ q^{-1/2} [B_{\tau(i)}, B_j]_q & \text{if } a_{ij} \neq -1 \text{ and } a_{\tau(i)j} = -1, \\ q^{-1} [B_i, [B_{\tau(i)}, B_j]_q] + B_j K_i K_{\tau(i)}^{-1} & \text{if } a_{ij} = -1 \text{ and } a_{\tau(i)j} = -1, \\ q^{-1} K_i K_{\tau(i)}^{-1} B_{\tau(i)} & \text{if } j = i, \\ q^{-1} K_{\tau(i)} K_i^{-1} B_i & \text{if } j = \tau(i), \\ B_j & \text{else.} \end{cases}$$

Observe that the case $a_{ij} = -1$ and $a_{\tau(i)j} = -1$ only occurs for $i = r - 1$ and $j = r$.

Theorem 4.3. *Let $1 \leq i \leq r - 1$.*

- 1) *There exists a unique algebra automorphism τ_i^- of $U'_q(\mathfrak{k})$ such that $\tau_i^-(B_j)$ is given by (4.5) for $j = 1, \dots, n$, and $\tau_i^-(x)$ for $x \in kT_\theta$ is given by (4.3).*
- 2) *The inverse automorphism τ_i of τ_i^- is determined by (4.3) and by*

$$(4.6) \quad \tau_i(B_j) = \begin{cases} q^{-1/2} [B_j, B_i]_q & \text{if } a_{ij} = -1 \text{ and } a_{\tau(i)j} \neq -1, \\ q^{-1/2} [B_j, B_{\tau(i)}]_q & \text{if } a_{ij} \neq -1 \text{ and } a_{\tau(i)j} = -1, \\ q^{-1} [[B_j, B_i]_q, B_{\tau(i)}]_q + B_j K_i K_{\tau(i)}^{-1} & \text{if } a_{ij} = -1 \text{ and } a_{\tau(i)j} = -1, \\ q K_{\tau(i)} K_i^{-1} B_{\tau(i)} & \text{if } j = i, \\ q K_i K_{\tau(i)}^{-1} B_i & \text{if } j = \tau(i), \\ B_j & \text{else.} \end{cases}$$

- 3) *The relation $\tau_{i-1} \tau_i \tau_{i-1} = \tau_i \tau_{i-1} \tau_i$ holds if $2 \leq i \leq r - 1$. Moreover, $\tau_i \tau_j = \tau_j \tau_i$ if $|i - j| \neq 1$.*

It remains to construct the algebra automorphisms τ_r^- and τ_r . To this end define

$$(4.7) \quad \tau_r^-(B_j) = \begin{cases} [B_r, B_j]_q & \text{if } j = r-1 \text{ or } j = r+1, \\ B_j & \text{else.} \end{cases}$$

Theorem 4.4. 1) *There exists a unique algebra automorphism τ_r^- of $U'_q(\mathfrak{k})$ such that $\tau_r^-(B_j)$ is given by (4.7) for $j = 1, \dots, n$, and $\tau_r^-(x)$ for $x \in kT_\theta$ is given by (4.4).*

2) *The inverse automorphism τ_r of τ_r^- is determined by (4.3) and by*

$$(4.8) \quad \tau_r(B_j) = \begin{cases} [B_j, B_r]_q & \text{if } j = r-1 \text{ or } j = r+1, \\ B_j & \text{else.} \end{cases}$$

3) *The relation $\tau_{-1}\tau_r\tau_{r-1}\tau_r = \tau_r\tau_{r-1}\tau_r\tau_{r-1}$ holds. Moreover, $\tau_r\tau_j = \tau_j\tau_r$ if $j = 1, \dots, r-2$.*

Summarizing the two above theorems one obtains the following result.

Corollary 4.5. *There exists a unique group homomorphism*

$$Br(\mathfrak{b}_r) \rightarrow \text{Aut}_{alg}(U'_q(\mathfrak{k}))$$

such that $\bar{s}_i \mapsto \tau_i$ for all $i = 1, \dots, r$.

4.2. The braid group action in the case (IIA) for even n . We now turn to the case (IIA) with $n = 2r$. Motivated by Proposition 2.2 and Corollary 2.3 we again aim to construct an action of the braid group $Br(\mathfrak{b}_r)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. As in (4.3) and (4.4) the restriction of this braid group action to the subalgebra kT_θ of $U'_q(\mathfrak{k})$ is determined by the Lusztig automorphisms. However, the difference between the cases $n = 2r-1$ and $n = 2r$ in Proposition 2.2 needs to be taken into account. More explicitly, for $i = 1, \dots, r-1$ and $x \in kT_\theta$ define

$$(4.9) \quad \tau_i(x) = \tau_i^-(x) = T_i T_{\tau(i)}(x),$$

$$(4.10) \quad \tau_r(x) = \tau_r^-(x) = T_r T_{r-1} T_r(x).$$

For $1 \leq i \leq r-1$ the automorphisms τ_i^- of $U'_q(\mathfrak{k})$ can be defined as in the previous subsection. They get simpler because the case ($a_{ij} = -1$ and $a_{\tau(i)j} = -1$) cannot occur if $n = 2r$. Hence, for $1 \leq i \leq r-1$ one defines

$$(4.11) \quad \tau_i^-(B_j) = \begin{cases} q^{-1/2}[B_i, B_j]_q & \text{if } a_{ij} = -1, \\ q^{-1/2}[B_{\tau(i)}, B_j]_q & \text{if } a_{\tau(i)j} = -1, \\ q^{-1}K_i K_{\tau(i)}^{-1} B_{\tau(i)} & \text{if } j = i, \\ q^{-1}K_{\tau(i)} K_i^{-1} B_i & \text{if } j = \tau(i), \\ B_j & \text{else.} \end{cases}$$

One obtains the following analog of Theorem 4.3.

Theorem 4.6. *Let $1 \leq i \leq r-1$.*

1) *There exists a unique algebra automorphism τ_i^- of $U'_q(\mathfrak{k})$ such that $\tau_i^-(B_j)$ is given by (4.11) for $j = 1, \dots, n$, and $\tau_i^-(x)$ for $x \in kT_\theta$ is given by (4.9).*

2) The inverse automorphism τ_i of τ_i^- is determined by (4.9) and by

$$(4.12) \quad \tau_i(B_j) = \begin{cases} q^{-1/2}[B_j, B_i]_q & \text{if } a_{ij} = -1, \\ q^{-1/2}[B_j, B_{\tau(i)}]_q & \text{if } a_{\tau(i)j} = -1, \\ qK_{\tau(i)}K_i^{-1}B_{\tau(i)} & \text{if } j = i, \\ qK_iK_{\tau(i)}^{-1}B_i & \text{if } j = \tau(i), \\ B_j & \text{else.} \end{cases}$$

3) The relation $\tau_{i-1}\tau_i\tau_{i-1} = \tau_i\tau_{i-1}\tau_i$ holds if $2 \leq i \leq r-1$. Moreover, $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| \neq 1$.

Again it remains to construct the algebra automorphisms τ_r^- and τ_r . Here the difference between the cases $n = 2r-1$ and $n = 2r$ in Proposition 2.2 is significant. Define

$$(4.13) \quad \tau_r^-(B_j) = \begin{cases} q^{-3/2}[B_{r+1}, [B_r, B_{r-1}]_q]_q + q^{1/2}K_r^{-1}K_{r+1}B_{r-1} & \text{if } j = r-1, \\ q^{-3/2}K_r^{-1}K_{r+1}B_r & \text{if } j = r, \\ q^{-3/2}K_rK_{r+1}^{-1}B_{r+1} & \text{if } j = r+1, \\ q^{-3/2}[B_r, [B_{r+1}, B_{r+2}]_q]_q + q^{1/2}K_rK_{r+1}^{-1}B_{r+2} & \text{if } j = r+2, \\ B_j & \text{else.} \end{cases}$$

Theorem 4.7. 1) There exists a unique algebra automorphism τ_r^- of $U'_q(\mathfrak{k})$ such that $\tau_r^-(B_j)$ is given by (4.13) for $j = 1, \dots, n$, and $\tau_r^-(x)$ for $x \in kT_\theta$ is given by (4.10).

2) The inverse automorphism τ_r of τ_r^- is determined by (4.9) and by

$$(4.14) \quad \tau_r(B_j) = \begin{cases} q^{-3/2}[[B_{r-1}, B_r]_q, B_{r+1}]_q + q^{-1/2}K_rK_{r+1}^{-1}B_{r-1} & \text{if } j = r-1, \\ q^{3/2}K_rK_{r+1}^{-1}B_r & \text{if } j = r, \\ q^{3/2}K_r^{-1}K_{r+1}B_{r+1} & \text{if } j = r+1, \\ q^{-3/2}[[B_{r+2}, B_{r+1}]_q, B_r]_q + q^{-1/2}K_r^{-1}K_{r+1}B_{r+2} & \text{if } j = r+2, \\ B_j & \text{else.} \end{cases}$$

3) The relation $\tau_{r-1}\tau_r\tau_{r-1}\tau_r = \tau_r\tau_{r-1}\tau_r\tau_{r-1}$ holds. Moreover, $\tau_r\tau_j = \tau_j\tau_r$ if $j = 1, \dots, r-2$.

Observe that Corollary 4.5 holds literally in the setting of the present subsection.

4.3. The braid group action in the case (IID). The braid group action on $U'_q(\mathfrak{k})$ in the cases (IID) and (IIE) are obtained as a combination of the results in the case (IIA) with $n = 2r-1$ odd with the results of Subsection 3. We first consider the case (IID), that is, \mathfrak{g} is of type D_{n+1} and $U'_q(\mathfrak{k})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$\begin{aligned} B_i &= F_i - K_i^{-1}E_i & \text{for } i = 1, \dots, n-1, \\ B_n &= F_n - K_n^{-1}E_{n+1}, \\ B_{n+1} &= F_{n+1} - K_{n+1}^{-1}E_n, \\ &K_nK_{n+1}^{-1}, K_n^{-1}K_{n+1}. \end{aligned}$$

Following Proposition 2.2 and Corollary 2.3 we aim to find an action of $Br(\mathfrak{b}_n)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. For $i = 1, \dots, n-1$, following the constructions from Subsections 3.1 and 4.1, define

$$(4.15) \quad \tau_i^-(B_j) = \begin{cases} [B_i, B_j]_q & \text{if } a_{ij} = -1, \\ B_j & \text{else.} \end{cases}$$

and for $x \in kT_\theta$ define moreover elements $\tau_i(x), \tau_i^-(x) \in kT_\theta$ by

$$(4.16) \quad \tau_i^-(x) = \tau_i(x) = T_i(x).$$

One obtains the following analog of Theorems 3.3 and 4.4.

Theorem 4.8. *Let $1 \leq i \leq n-1$.*

- 1) *There exists a unique algebra automorphism τ_i^- of $U'_q(\mathfrak{k})$ such that $\tau_i^-(B_j)$ is given by (4.15) for $j = 1, \dots, n+1$, and $\tau_i^-(x)$ for $x \in kT_\theta$ is given by (4.16).*
- 2) *The inverse automorphism τ_i of τ_i^- is determined by (4.16) and by*

$$(4.17) \quad \tau_i(B_j) = \begin{cases} [B_j, B_i]_q & \text{if } a_{ij} = -1, \\ B_j & \text{else.} \end{cases}$$

- 3) *The relation $\tau_{i-1}\tau_i\tau_{i-1} = \tau_i\tau_{i-1}\tau_i$ holds if $2 \leq i \leq n-1$. Moreover, $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| \neq 1$ and $1 \leq j \leq n-1$.*

Now we follow (4.5) and define

$$(4.18) \quad \tau_n^-(B_j) = \begin{cases} q^{-1}[B_n, [B_{n+1}, B_{n-1}]_q]_q + B_{n-1}K_nK_{n+1}^{-1} & \text{if } j = n-1, \\ q^{-1}K_nK_{n+1}^{-1}B_{n+1} & \text{if } j = n, \\ q^{-1}K_{n+1}K_n^{-1}B_n & \text{if } j = n+1, \\ B_j & \text{else} \end{cases}$$

and

$$(4.19) \quad \tau_n^-(x) = \tau_n(x) = T_nT_{n+1}(x)$$

for all $x \in kT_\theta$.

Theorem 4.9. 1) *There exists a unique algebra automorphism τ_n^- of $U'_q(\mathfrak{k})$ such that $\tau_n^-(B_j)$ is given by (4.18) for $j = 1, \dots, n+1$, and $\tau_n^-(x)$ for $x \in kT_\theta$ is given by (4.19).*

- 2) *The inverse automorphism τ_n of τ_n^- is determined by (4.19) and by*

$$(4.20) \quad \tau_n(B_j) = \begin{cases} q^{-1}[[B_{n-1}, B_n]_q, B_{n+1}]_q + B_{n-1}K_nK_{n+1}^{-1} & \text{if } j = n-1, \\ qK_{n+1}K_n^{-1}B_{n+1} & \text{if } j = n, \\ qK_nK_{n+1}^{-1}B_n & \text{if } j = n+1, \\ B_j & \text{else.} \end{cases}$$

- 3) *The relation $\tau_{n-1}\tau_n\tau_{n-1}\tau_n = \tau_n\tau_{n-1}\tau_n\tau_{n-1}$ holds. Moreover, $\tau_n\tau_j = \tau_j\tau_n$ if $j = 1, \dots, n-2$.*

Theorems 4.8 and 4.9 yield an action of the braid group $Br(\mathfrak{b}_n)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms such that $\bar{s}_i \mapsto \tau_i$ for $i = 1, \dots, n$.

4.4. The braid group action in the case (IIE). We now turn to the case (IIE), that is, \mathfrak{g} is of type E_6 and $U'_q(\mathfrak{k})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$\begin{aligned} B_1 &= F_1 - K_1^{-1}E_6, & B_4 &= F_4 - K_4^{-1}E_4, \\ B_2 &= F_2 - K_2^{-1}E_2, & B_5 &= F_5 - K_5^{-1}E_3, \\ B_3 &= F_3 - K_3^{-1}E_5, & B_6 &= F_6 - K_6^{-1}E_1, \\ K_1K_6^{-1}, K_6^{-1}K_1, & & K_3K_5^{-1}, K_5^{-1}K_3. \end{aligned}$$

By Proposition 2.2 and Corollary 2.3 we expect to find an action of $Br(\mathfrak{f}_4)$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. For $x \in kT_\theta$ and $j = 1, 2, \dots, 6$ define

$$(4.21) \quad \tau_1(x) = T_1T_6(x), \quad \tau_1(B_j) = \begin{cases} qK_6K_1^{-1}B_6 & \text{if } j = 1, \\ B_j & \text{if } j = 2, 4, \\ q^{-1/2}[B_3, B_1]_q & \text{if } j = 3, \\ q^{-1/2}[B_5, B_6]_q & \text{if } j = 5, \\ qK_1K_6^{-1}B_1 & \text{if } j = 6, \end{cases}$$

$$(4.22) \quad \tau_2(x) = T_3T_5(x), \quad \tau_2(B_j) = \begin{cases} q^{-1/2}[B_1, B_3]_q & \text{if } j = 1, \\ B_2 & \text{if } j = 2, \\ qK_5K_3^{-1}B_5 & \text{if } j = 3, \\ q^{-1}[[B_4, B_3]_q, B_5]_q + B_4K_3K_5^{-1} & \text{if } j = 4, \\ qK_3K_5^{-1}B_3 & \text{if } j = 5, \\ q^{-1/2}[B_6, B_5]_q & \text{if } j = 6, \end{cases}$$

$$(4.23) \quad \tau_3(x) = T_4(x), \quad \tau_3(B_j) = \begin{cases} B_j & \text{if } j = 1, 4, 6, \\ [B_j, B_4]_q & \text{if } j = 2, 3, 5, \end{cases}$$

$$(4.24) \quad \tau_4(x) = T_2(x), \quad \tau_4(B_j) = \begin{cases} B_j & \text{if } j = 1, 2, 3, 5, 6, \\ [B_3, B_2] & \text{if } j = 4. \end{cases}$$

The next theorem is implied by the corresponding results in the cases (IIA) for odd n and (IID) from Subsections 4.1 and 4.3, respectively.

Theorem 4.10. 1) *There exist uniquely determined algebra automorphisms τ_i of $U'_q(\mathfrak{k})$ for $i = 1, \dots, 4$ such that $\tau_i(B_j)$ and $\tau_i(x)$ are given by (4.21) – (4.24) for $j = 1, \dots, 6$ and $x \in kT_\theta$.*
 2) *There exists a unique group homomorphism $Br(\mathfrak{f}_4) \rightarrow \text{Aut}_{alg}(U'_q(\mathfrak{k}))$ such that $\bar{s}_i \mapsto \tau_i$ for all $i = 1, \dots, 4$.*

The inverse automorphisms τ_i^- of τ_i for $i = 1, \dots, 4$ can also be read off the corresponding formulas in Subsections 4.1 and 4.3.

4.5. The proofs of the theorems in Section 4. The proofs of the theorems in Subsections 4.1, 4.2, and 4.3 are again performed via **GAP** using the codes **II-A7.txt**, **II-A6.txt**, and **II-D5.txt**, respectively, which are available from [WWW].

We first turn to the case (IIA) for odd n considered in Subsection 4.1. Observe that it suffices to prove Theorems 4.3 and 4.4 in the case $r = 4$, that is for $n = 7$. Indeed, if in this case τ_2 and τ_2^- are mutually inverse algebra automorphisms of

$U'_q(\mathfrak{k})$ then τ_i and τ_i^- are mutually inverse algebra automorphisms of $U'_q(\mathfrak{k})$ for $i = 1, \dots, r-2$ in the case of arbitrary r . Similarly, the fact that for $r = 4$ the maps τ_3 and τ_4 are algebra automorphisms with inverses τ_3^- and τ_4^- , respectively, implies that for general r the maps τ_{r-1} and τ_r are algebra automorphisms with inverses τ_{r-1}^- and τ_r^- , respectively. Finally, the braid relations between the τ_i for $r = 4$ imply the braid relations between the τ_i for general r . With the GAP-code `II-A7.txt` one checks all the relations necessary to prove Theorems 4.3 and 4.4 by a method similar to the one described in Subsection 3.2 for case I.

In the case (IIA) for even n one sees by reasoning similar to the one above that it is sufficient to prove the theorems of Subsection 4.2 in the case $r = 3$, that is $n = 6$. With the GAP-code `II-A6.txt` one may check all the necessary relations in this case. Similarly, it suffices to consider the case that \mathfrak{g} is of type D_5 in order to verify the results of Subsection 4.3. The GAP-code `II-D5.txt` performs all the necessary checks.

5. THE INVOLUTIVE AUTOMORPHISM $\text{Ad}(w_X) \circ \omega$

In this section we consider the case (III), that is, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ with $n = 2m$ even, and $\theta = \text{Ad}(w_X) \circ \omega$ where $w_X = s_1 s_3 s_5 \cdots s_{2m-1}$. In this case, by definition, $U'_q(\mathfrak{k})$ is the subalgebra of $U_q(\mathfrak{sl}_n(\mathbb{C}))$, generated by the elements

$$(5.1) \quad E_i, F_i, K_i^{\pm 1} \quad \text{for } i \text{ odd,}$$

$$(5.2) \quad F_i - K_i^{-1} \text{ad}(E_{i-1} E_{i+1})(E_i) \quad \text{for } i \text{ even.}$$

Here $\text{ad}(x)(u) = \sum_r x_r u S(x'_r)$ denotes the adjoint action of x on u for $u, x \in U_q(\mathfrak{g})$ with $\Delta(x) = \sum_r x_r \otimes x'_r$, see [Jan96, 4.18]. Again one checks that $U'_q(\mathfrak{k})$ is a right coideal subalgebra of $U_q(\mathfrak{sl}_n(\mathbb{C}))$. We define

$$B_i = \begin{cases} F_i & \text{if } i \text{ is odd,} \\ F_i - K_i^{-1} \text{ad}(E_{i-1} E_{i+1})(E_i) & \text{if } i \text{ is even.} \end{cases}$$

Remark 5.1. Let $T_{w_X} = T_1 T_3 \dots T_{n-1}$ denote the Lusztig automorphism corresponding to the element $w_X = s_1 s_3 \dots s_{n-1} \in W$. The generators (5.2) can be written as $F_i - K_i^{-1} T_{w_X}(E_i)$. This shows for even i that B_i is a q -analog of $f_i + \theta(f_i)$.

From [Let03, Theorem 7.1] we know how to write $U'_q(\mathfrak{k})$ in terms of generators and relations. Let $\mathcal{M}^{\geq 0}$ denote the subalgebra of $U'_q(\mathfrak{k})$ generated by the elements of the set $\{E_i, K_i, K_i^{-1} \mid i \text{ odd}\}$.

Proposition 5.2. *The algebra $U'_q(\mathfrak{k})$ is generated over $\mathcal{M}^{\geq 0}$ by elements B_i , $1 \leq i \leq n-1$, subject only to the following relations:*

- (1) $K_i B_j K_i^{-1} = q^{a_{ij}} B_j$ for $1 \leq i, j \leq n-1$ with i odd,
- (2) $E_i B_j - B_j E_i = \delta_{ij} (K_i - K_i^{-1}) / (q - q^{-1})$ for $1 \leq i, j \leq n-1$ with i odd,
- (3) $B_i B_j - B_j B_i = 0$ if $a_{ij} = 0$,
- (4) $B_i^2 B_j - (q + q^{-1}) B_i B_j B_i + B_j B_i^2 = 0$ if $a_{ij} = -1$ and i odd.
- (5) If $a_{ij} = -1$ and i even then

$$\begin{aligned} & B_i^2 B_j - (q + q^{-1}) B_i B_j B_i + B_j B_i^2 \\ & = -q^{-1} ((q - q^{-1})^2 F_j E_j E_{j|i} + (q^{-1} K_j^{-1} + q K_j) E_{j|i}) \end{aligned}$$

where

$$j|i = \begin{cases} i+1 & \text{if } j = i-1, \\ i-1 & \text{if } j = i+1. \end{cases}$$

Remark 5.3. In [Nou96] Noumi defined a subalgebra $U_q^{\text{tw}}(\mathfrak{k})$ of $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_{2m})$ which depends on an explicit solution of the reflection equation. This subalgebra is generated by the elements of a matrix K given in [Nou96, (2.19)], [NS95, 3.6]. The properties of the matrices L^+ and L^- occurring in the former reference imply that $U_q^{\text{tw}}(\mathfrak{k})$ is a right coideal subalgebra of $U_q(\mathfrak{g})$ [Nou96, Section 2.4], [Let99, Lemma 6.3]. As stated in [Let99, Remark 6.7] one can show by direct computation that $U'_q(\mathfrak{k}) = U_q^{\text{tw}}(\mathfrak{k})$ for $m = 2$. This implies, for general m , that all generators of $U'_q(\mathfrak{k})$ are contained in $U_q^{\text{tw}}(\mathfrak{k})$ and hence $U'_q(\mathfrak{k}) \subseteq U_q^{\text{tw}}(\mathfrak{k})$. On the other hand it was proved in [Let02] that $U'_q(\mathfrak{k})$ is a maximal right coideal subalgebra specializing to $U(\mathfrak{k})$ in a suitable limit $q \rightarrow 1$. This implies, at least for generic q , that $U_q^{\text{tw}}(\mathfrak{k}) = U'_q(\mathfrak{k})$. In particular, up to notational changes, the algebra $U'_q(\mathfrak{k})$ as defined above coincides with the algebra $U'_q(\mathfrak{k})$ considered in [MR08].

5.1. Braid group action. By Corollary 2.3 one expects an action of $Br(\mathfrak{a}_{m-1})$ on $U'_q(\mathfrak{k})$ by algebra automorphisms. Moreover, the generators \overline{s}_i , for $i = 1, \dots, m-1$ of $Br(\mathfrak{a}_{m-1})$ should act in highest degree as $T_{2i}T_{2i-1}T_{2i+1}T_{2i}$ or its inverse. Observe that

$$(5.3) \quad T_{2i}^{-1}T_{2i-1}^{-1}T_{2i+1}^{-1}T_{2i}^{-1}(F_j) = \begin{cases} [[F_{2i}, F_{2i-1}]_q, F_{2i-2}]_q & \text{if } j = 2i-2, \\ F_{2i+1} & \text{if } j = 2i-1, \\ F_{2i-1} & \text{if } j = 2i+1, \\ [[F_{2i}, F_{2i+1}]_q, F_{2i+2}]_q & \text{if } j = 2i+2 \end{cases}$$

and

$$(5.4) \quad \begin{aligned} & T_{2i}^{-1}T_{2i-1}^{-1}T_{2i+1}^{-1}T_{2i}^{-1}(-K_{2i}^{-1}T_{w_X}(E_{2i})) \\ &= -K_{2i-1}K_{2i}K_{2i+1}T_{2i}^{-1}T_{2i-1}^{-1}T_{2i+1}^{-1}T_{2i}^{-1}T_{2i-1}T_{2i+1}(E_{2i}) \\ &= -K_{2i-1}K_{2i}K_{2i+1}T_{2i-1}T_{2i+1}T_{2i}^{-1}T_{2i-1}^{-1}T_{2i+1}^{-1}T_{2i}^{-1}(E_{2i}) \\ &= K_{2i-1}K_{2i+1}T_{2i-1}^2T_{2i+1}^2(F_{2i}) \\ &= (q - q^{-1})^2[[F_{2i}, F_{2i+1}]_q, F_{2i-1}]_q E_{2i-1}E_{2i+1} \\ &\quad - q^2(q - q^{-1})[F_{2i}, F_{2i+1}]_q K_{2i-1}E_{2i+1} \\ &\quad - q^2(q - q^{-1})[F_{2i}, F_{2i-1}]_q K_{2i+1}E_{2i-1} + q^4 F_{2i}K_{2i-1}K_{2i+1}. \end{aligned}$$

Equation (5.3) motivates the following definition. For $i = 1, \dots, m-1$ and $j = 1, 3, 5, \dots, 2m-1$ define

$$(5.5) \quad \tau_i^-(X_j) = \begin{cases} X_{2i+1} & \text{if } j = 2i-1, \\ X_{2i-1} & \text{if } j = 2i+1, \\ X_j & \text{if } j \text{ is odd and } j \notin \{2i-1, 2i+1\} \end{cases}$$

where X denotes any of the symbols F, E, K , or K^{-1} . Moreover, Equations (5.3) and (5.4) suggest the following definition up to the insertion of additional q -factors.

For $j \neq 2i$ even define

$$(5.6) \quad \tau_i^-(B_j) = \begin{cases} q^{-1/2}[[B_{2i}, B_{2i-1}]_q, B_{2i-2}]_q & \text{if } j = 2i - 2, \\ q^{-1/2}[[B_{2i}, B_{2i+1}]_q, B_{2i+2}]_q & \text{if } j = 2i + 2, \\ B_j & \text{if } j \notin \{2i - 2, 2i, 2i + 2\}, \end{cases}$$

and in the case $j = 2i$ define

$$(5.7) \quad \begin{aligned} \tau_i^-(B_{2i}) &= q^{-1}(q - q^{-1})^2[[B_{2i}, B_{2i+1}]_q, B_{2i-1}]_q E_{2i-1} E_{2i+1} \\ &\quad - q(q - q^{-1})[B_{2i}, B_{2i+1}]_q K_{2i-1} E_{2i+1} \\ &\quad - q(q - q^{-1})[B_{2i}, B_{2i-1}]_q K_{2i+1} E_{2i-1} + q^3 B_{2i} K_{2i-1} K_{2i+1}. \end{aligned}$$

Theorem 5.4. *Let $1 \leq i \leq m - 1$.*

- 1) *There exists a unique algebra automorphism τ_i^- of $U'_q(\mathfrak{k})$ such that $\tau_i^-(B_j)$ is given by (5.6) and (5.7) for even j with $2 \leq j \leq 2m - 2$, and $\tau_i^-(X_j)$ is given by (5.5) for odd j with $1 \leq j \leq 2m - 1$.*
- 2) *The inverse automorphism τ_i of τ_i^- is determined by $\tau_i(X_j) = \tau_i^-(X_j)$ for odd j with $1 \leq j \leq 2m - 1$, where the symbol X denotes any of the symbols E, F, K , and K^{-1} , and by*

$$(5.8) \quad \tau_i(B_j) = \begin{cases} q^{-1/2}[[B_{2i-2}, B_{2i-1}]_q, B_{2i}]_q & \text{if } j = 2i - 2, \\ q^{-1/2}[[B_{2i+2}, B_{2i+1}]_q, B_{2i}]_q & \text{if } j = 2i + 2, \\ B_j & \text{if } j \notin \{2i - 2, 2i, 2i + 2\}, j \text{ even}, \end{cases}$$

$$(5.9) \quad \begin{aligned} \tau_i(B_{2i}) &= q^{-1}(q - q^{-1})^2[B_{2i+1}, [B_{2i-1}, B_{2i}]_q]_q E_{2i-1} E_{2i+1} \\ &\quad - q^{-2}(q - q^{-1})[B_{2i+1}, B_{2i}]_q K_{2i-1} E_{2i+1} \\ &\quad - q^{-2}(q - q^{-1})[B_{2i-1}, B_{2i}]_q K_{2i+1} E_{2i-1} + q^{-3} B_{2i} K_{2i-1}^{-1} K_{2i+1}^{-1}. \end{aligned}$$

- 3) *The relation $\tau_{i-1}\tau_i\tau_{i-1} = \tau_i\tau_{i-1}\tau_i$ holds if $2 \leq i \leq m - 1$. Moreover, $\tau_i\tau_j = \tau_j\tau_i$ if $|i - j| \neq 1$ and $1 \leq j \leq m - 1$.*

It suffices to verify Theorem 5.4 in the case where $\mathfrak{g} = \mathfrak{sl}_8(\mathbb{C})$. In this case all necessary checks are performed by the GAP-code `III-A7.txt` which is available from [WWW].

Remark 5.5. If one sets q and K_j equal to 1 in Formulas (5.8) and (5.9) then one obtains precisely Formula (2.7) from the classical case. It is in this sense that the action of τ_i on $U'_q(\mathfrak{k})$ is a deformation of the action of $\text{Ad}(s_{2i}s_{2i-1}s_{2i+1}s_{2i})$ on $U(\mathfrak{sp}_{2m}(\mathbb{C}))$. Most interesting are the higher degree terms in (5.7) and (5.9) which we were only able to find by comparison with the Lusztig action given by (5.4).

5.2. Action of $\mathbb{Z}^m \rtimes Br(\mathfrak{a}_{m-1})$ on $U'_q(\mathfrak{k})$. We now combine the action of $Br(\mathfrak{a}_{m-1})$ on $U'_q(\mathfrak{k})$ constructed in the previous subsection with the action of \mathbb{Z}^m on $U'_q(\mathfrak{k})$ given by the Lusztig automorphisms T_i for $i = 1, 3, \dots, 2m - 1$. In view of Remarks 5.3 and 5.5 this will provide a proof of Theorem 1.1.

Lemma 5.6. *Let $j \in \{1, \dots, 2m - 1\}$ be odd. Then $T_j(U'_q(\mathfrak{k})) = U'_q(\mathfrak{k})$.*

Proof. It suffices to prove that $T_j^{-1}(U'_q(\mathfrak{k})) = U'_q(\mathfrak{k})$. The Lusztig automorphism T_j^{-1} maps the generators (5.1) to $U'_q(\mathfrak{k})$. Hence it remains to show that $T_j^{-1}(B_i) \in U'_q(\mathfrak{k})$

if $|i - j| = 1$. To this end assume that $j = i - 1$ and calculate

$$\begin{aligned}
& F_{i-1}K_i^{-1}\text{ad}(E_{i-1}E_{i+1})(E_i) - qK_i^{-1}\text{ad}(E_{i-1}E_{i+1})(E_i)F_{i-1} \\
&= qK_i^{-1}(F_{i-1}\text{ad}(E_{i-1}E_{i+1})(E_i) - \text{ad}(E_{i-1}E_{i+1})(E_i)F_{i-1}) \\
&= qK_i^{-1}\text{ad}(F_{i-1})(\text{ad}(E_{i-1}E_{i+1})(E_i))K_{i-1}^{-1} \\
&= K_{i-1}^{-1}K_i^{-1}\text{ad}(E_{i+1})E_i \\
&= T_{i-1}^{-1}(K_i^{-1}T_{i-1}T_{i+1}(E_i)).
\end{aligned}$$

Hence one obtains

$$\begin{aligned}
T_{i-1}^{-1}(B_i) &= T_{i-1}^{-1}(F_i) - T_{i-1}^{-1}(K_i^{-1}T_{i-1}T_{i+1}(E_i)) \\
&= F_{i-1}B_i - qB_iF_{i-1}
\end{aligned}$$

and this element does belong to $U'_q(\mathfrak{k})$. \square

The Lusztig automorphisms T_j for $j = 1, 3, 5, \dots, 2m - 1$ commute pairwise. Hence the above lemma produces an action of the additive group \mathbb{Z}^m on $U'_q(\mathfrak{k})$ by algebra automorphisms. The braid group $Br(\mathfrak{a}_{m-1})$ acts on \mathbb{Z}^m from the left and one may hence form the semidirect product $\mathbb{Z}^m \rtimes Br(\mathfrak{a}_{m-1})$ as follows. Let e_i , $i = 1, \dots, m$, denote the standard basis of \mathbb{Z}^m . Then the multiplication on $\mathbb{Z}^m \rtimes Br(\mathfrak{a}_{m-1})$ is determined by

$$(5.10) \quad (e_i, \sigma) \cdot (e_{i'}, \mu) = (e_i + e_{\pi(\sigma)(i')}, \sigma \circ \mu)$$

where $\pi : Br(\mathfrak{a}_{m-1}) \rightarrow S_m$ denotes the canonical projection onto the symmetric group in m elements. By the following theorem the actions of \mathbb{Z}^m and $Br(\mathfrak{a}_{m-1})$ on $U'_q(\mathfrak{k})$ give the desired action of $\mathbb{Z}^m \rtimes Br(\mathfrak{a}_{m-1})$.

Theorem 5.7. *There exists a unique group homomorphism*

$$\mathbb{Z}^m \rtimes Br(\mathfrak{a}_{m-1}) \rightarrow \text{Aut}_{alg}(U'_q(\mathfrak{k}))$$

such that $(e_i, 1) \mapsto T_{2i-1}$ for $i = 1, \dots, m$, and $(0, \overline{s}_j) \mapsto \tau_j$ for $j = 1, \dots, m - 1$.

Proof. To prove the theorem one needs to verify the following equalities

$$(5.11) \quad \tau_i \circ T_{2i-1} = T_{2i+1} \circ \tau_i, \quad \tau_i \circ T_{2i+1} = T_{2i-1} \circ \tau_i \quad \text{for } i = 1, \dots, m-1,$$

$$(5.12) \quad \tau_j \circ T_{2i-1} = T_{2i-1} \circ \tau_j \quad \text{if } j \neq i, i-1$$

of automorphisms of $U'_q(\mathfrak{k})$. Again, it suffices to verify these equations on the generators of $U'_q(\mathfrak{k})$ as both sides of the equations are already known to be algebra automorphisms. It is straightforward to check Equations (5.11) and (5.12) when evaluated on the elements $\{E_j, F_j, K_j^{\pm 1} \mid j \text{ odd}\}$. To obtain Equation (5.12) it hence suffices to check that

$$\tau_i \circ T_{2i-3}^{-1}(B_{2i-2}) = T_{2i-3}^{-1} \circ \tau_i(B_{2i-2}), \quad \tau_j \circ T_{2j+3}^{-1}(B_{2j+2}) = T_{2j+3}^{-1} \circ \tau_j(B_{2j+2})$$

for $i = 2, \dots, m - 1$ and $j = 1, \dots, m - 2$. This follows from (5.8) and from the relations $T_{2i-3}^{-1}(B_{2i-2}) = [B_{2i-3}, B_{2i-2}]_q$ and $T_{2i+3}^{-1}(B_{2i+2}) = [B_{2i+3}, B_{2i+2}]_q$ which were verified in the proof of Lemma 5.6.

For symmetry reasons it now suffices to verify the first equation of (5.11). To complete the proof of the theorem we hence have to show that

$$(5.13) \quad \tau_i \circ T_{2i-1}^{-1}(B_{2i-2}) = T_{2i+1}^{-1} \circ \tau_i(B_{2i-2}),$$

$$(5.14) \quad \tau_i \circ T_{2i-1}^{-1}(B_{2i}) = T_{2i+1}^{-1} \circ \tau_i(B_{2i}),$$

$$(5.15) \quad \tau_i \circ T_{2i-1}^{-1}(B_{2i+2}) = T_{2i+1}^{-1} \circ \tau_i(B_{2i+2}).$$

Equation (5.13) follows from $T_{2i-1}^{-1}(B_{2i-2}) = [B_{2i-1}, B_{2i-2}]_q$ and $T_{2i+1}^{-1}(B_{2i}) = [B_{2i+1}, B_{2i}]_q$ and the definition of τ_i in Theorem 5.4.2). Equations (5.14) and (5.15) are equivalent to

$$(5.16) \quad [\tau_i(B_{2i-1}), \tau_i(B_{2i})]_q = T_{2i+1}^{-1}(\tau_i(B_{2i})),$$

$$(5.17) \quad \tau_i(B_{2i+2}) = T_{2i+1}^{-1}(\tau_i(B_{2i+2})),$$

respectively, which are checked by computer calculations at the end of the file `III-A7.txt`. This concludes the proof of the Theorem. \square

REFERENCES

- [AK] J. Apel and U. Klaus, *FELIX*, (<http://felix.hgb-leipzig.de>).
- [Ara62] S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.
- [Bou02] N. Bourbaki, *Lie groups and Lie algebras, Chapters 4–6*, Springer-Verlag, Berlin, 2002.
- [Che07] L.O. Chekhov, *Teichmüller theory of bordered surfaces*, SIGMA Symmetry and Integrability Geom. Methods Appl. **3** (2007), Paper 066, 37 pp.
- [CKP95] C. De Concini, V.G. Kac, and C. Procesi, *Some quantum analogues of solvable Lie groups*, Geometry and analysis (Bombay 1992) (Bombay), Tata Inst. Fund. Res., 1995, pp. 41–65.
- [dG07] W.A. de Graaf, *GAP package QuaGroup – a package for doing computations with quantum groups*, version 1.3, (<http://www.gap-system.org/Packages/quagroup.html>) (2007).
- [Dij96] M.S. Dijkhuizen, *Some remarks on the construction of quantum symmetric spaces*, Acta Appl. Math. **44** (1996), no. 1–2, 59–80.
- [GAP08] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.
- [GK91] A.M. Gavrilik and A.U. Klimyk, *q-deformed orthogonal and pseudo-orthogonal algebras and their representations*, Lett. Math. Phys. **21** (1991), no. 3, 215–220.
- [HS09] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Preprint, [arXiv:0909.0293](https://arxiv.org/abs/0909.0293) (2009), 43 pp.
- [Jan96] J.C. Jantzen, *Lectures on quantum groups*, Grad. Stud. Math., vol. 6, Amer. Math. Soc, Providence, RI, 1996.
- [Kol08] S. Kolb, *Quantum symmetric pairs and the reflection equation*, Algebr. Represent. Theory **11** (2008), no. 6, 519–544.
- [Let99] G. Letzter, *Symmetric pairs for quantized enveloping algebras*, J. Algebra **220** (1999), 729–767.
- [Let02] ———, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras (Cambridge), MSRI publications, vol. 43, Cambridge Univ. Press, 2002, pp. 117–166.
- [Let03] ———, *Quantum symmetric pairs and their zonal spherical functions*, Transformation Groups **8** (2003), 261–292.
- [Lus90] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–114.
- [Lus93] ———, *Introduction to quantum groups*, Progress in Mathematics, no. 110, Birkhäuser, Boston, MA, 1993.
- [Lus96] ———, *Braid group action and canonical bases*, Adv. Math. **122** (1996), no. 2, 237–261.
- [MR08] A.I. Molev and E. Ragoucy, *Symmetries and invariants of twisted quantum algebras and associated Poisson algebras*, Rev. Math. Phys. **20** (2008), no. 2, 173–198.
- [Nou96] M. Noumi, *Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123** (1996), 16–77.

- [NS95] M. Noumi and T. Sugitani, *Quantum symmetric spaces and related q -orthogonal polynomials*, Group theoretical methods in physics (Singapore) (A. Arima et. al., ed.), World Scientific, 1995, pp. 28–40.
- [Pel10] J. Pellegrini, *Braid group actions on coideal subalgebras of quantized enveloping algebras*, Master's thesis, Università degli Studi di Padova, 2010.
- [Ste68] R. Steinberg, *Lectures on Chevalley groups. Notes prepared by John Faulkner and Robert Wilson*, Yale University, New Haven, Conn., 1968.
- [WWW] <http://www.staff.ncl.ac.uk/stefan.kolb/GAPfiles.html>.

STEFAN KOLB, SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE NE1 7RU, UK

E-mail address: `stefan.kolb@ncl.ac.uk`

JACOPO PELLEGRINI, DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY

E-mail address: `pellej14@hotmail.com`